Linear Algebra and its Applications 436 (2012) 753–757

Contents lists available at SciVerse ScienceDirect

Linear Algebra and its Applications

journal homepage: www.elsevier.com/locate/laa

The maximal length of a chain in the Bruhat order for a class of binary matrices ☆

Alessandro Conflitti [a], C.M. da Fonseca [b,*], Ricardo Mamede [b]

[a] CMUC, Centre for Mathematics, University of Coimbra, Apartado 3008, 3001-454 Coimbra, Portugal
[b] Department of Mathematics, University of Coimbra, 3001-454 Coimbra, Portugal

ARTICLE INFO

Article history:
Received 24 May 2011
Accepted 25 July 2011
Available online 25 August 2011

Submitted by R.A. Brualdi

AMS classification:
05B20
06A07
15A36

Keywords:
Bruhat order
Row and column sum vectors
(0, 1)-Matrices
Interchange
Minimal matrix
Maximal matrix

ABSTRACT

We answer a question by Brualdi and Deaett about the maximal length of a chain in the Bruhat order for an interesting combinatorial class of binary matrices.

1. Overview and definitions

Let m and n be two positive integers and let $R = (r_1, \ldots, r_m)$ and $S = (s_1, \ldots, s_n)$ be positive integral vectors. The class of all m-by-n $(0, 1)$-matrices with row sum vector R and column sum vector S is denoted by $\mathscr{A}(R, S)$. Combinatorial properties of such class of matrices have been thoroughly explored over the years (cf. e.g. [2–8,10,11,15,17] and references therein). The Gale–Ryser Theorem, independently due to Gale [12] and Ryser [17], states that a necessary and sufficient condition for

☆ This work is supported by CMUC – Centro de Matemática da Universidade de Coimbra.
* Corresponding author.
E-mail addresses: conflitt@mat.uc.pt (A. Conflitti), cmf@mat.uc.pt (C.M. da Fonseca), mamede@mat.uc.pt (R. Mamede).

0024-3795/$ - see front matter © 2011 Elsevier Inc. All rights reserved.
doi:10.1016/j.laa.2011.07.043

$\mathscr{A}(R, S)$ to be nonempty is $S \trianglelefteq R^*$, where R^* is the conjugate of R and $\trianglelefteq$ is the standard majorization order. We are assuming that R and S are both nonincreasing vectors. An important case in which nonemptiness is assured occurs when $m = n$, k is a positive integer number such that $0 \leq k \leq n$, and $R = S = (k, \ldots, k)$ is the constant vector having each component equal to k. In this case we write $\mathscr{A}(n, k)$ instead of $\mathscr{A}(R, S)$.

In [8] a Bruhat partial order $\preccurlyeq$ on a nonempty class $\mathscr{A}(R, S)$ was defined using a characterization of the Bruhat order on S_n, the symmetric group of n elements, seen as the set of permutation matrices $\mathscr{A}(n, 1)$. Precisely, for an $m \times n$ matrix $A = (a_{ij})$, let $\Sigma_A = (\sigma_{ij}(A))$ be the $m \times n$ matrix defined by

$$\sigma_{ij}(A) = \sum_{k=1}^{i} \sum_{\ell=1}^{j} a_{k\ell}, \quad \text{for } 1 \leq i \leq m, \quad 1 \leq j \leq n.$$

If $A_1, A_2 \in \mathscr{A}(R, S)$, then $A_1 \preccurlyeq A_2$ if and only if $\Sigma_{A_1} \geq \Sigma_{A_2}$ in the entrywise order, i.e., $\sigma_{ij}(A_1) \geq \sigma_{ij}(A_2)$, for all $1 \leq i \leq m$ and $1 \leq j \leq n$.

In [7], Brualdi and Deaett characterize all families of the class $\mathscr{A}(n, k)$ with a unique minimal element.

Theorem 1.1 [7, Theorem 5.1]. *Let $n \geq 1$ and k be integers with $0 \leq k \leq n$. Then the class $\mathscr{A}(n, k)$ has a unique minimal element in the Bruhat order if and only if $k \in \{0, 1, n-1, n\}$ or $n = 2k$. Moreover the minimal matrix in $\mathscr{A}(2k, k)$ is*

$$P_k = J_k \oplus J_k = \begin{pmatrix} J_k & O_k \\ O_k & J_k \end{pmatrix},$$

where J_k is the matrix of all 1's of order k and O_k is the zero matrix also of order k.

Since $\mathscr{A}(n, k) \simeq \mathscr{A}(n, n-k)$ (the map $A \longmapsto J_n - A$ does the job), $|\mathscr{A}(n, 0)| = 1$, and $\mathscr{A}(n, 1) \simeq S_n$, the most interesting case is in fact $\mathscr{A}(2k, k)$.

We remark that $|\mathscr{A}(2k, k)|$ is the sequence A058527 in the *The On-Line Encyclopedia of Integer Sequences* [18]. For instances of values of such sequence the reader is referred to [19]. We observe also that computing a closed formula for such sequence is an open problem which looks quite hard (see e.g. [1,9,13,14,16] and the references therein for some asymptotic results).

In [7, Section 6] an example is provided to show that Bruhat order $\preccurlyeq$ is not graded, and it is wondered if the maximal length of a chain in the Bruhat order in the class $\mathscr{A}(2k, k)$ from the minimal element P_k to the maximal element

$$Q_k = \begin{pmatrix} O_k & J_k \\ J_k & O_k \end{pmatrix}$$

is $4k^2$.

In this paper we show that the correct answer to the above question is a much larger number: namely k^4.

2. The main result

For any $A, B \in \mathscr{A}(R, S)$ such that $A \preccurlyeq B$, as an immediate consequence of the definition of Bruhat order, an upper bound for the length of any admissible chain between A and B is clearly given by

$$\varphi(A, B) = \sum_{i=1}^{m} \sum_{j=1}^{n} \left[\sigma_{ij}(A) - \sigma_{ij}(B)\right].$$

Since the poset $(\mathscr{A}(2k, k), \preccurlyeq)$ admits a unique minimum P_k and a unique maximum Q_k, any chain between two pairwise comparable elements can be extended to a chain between P_k and Q_k.

After some lengthy but rather straightforward computations we get

$$\sigma_{ij}(P_k) = \begin{cases} ij & \text{if } i, j \leq k, \\ ik & \text{if } i \leq k \leq j, \\ jk & \text{if } i \geq k \geq j, \\ ij - k(i + j - 2k) & \text{if } i, j \geq k, \end{cases}$$

$$\sigma_{ij}(Q_k) = \begin{cases} 0 & \text{if } i, j \leq k, \\ i(j - k) & \text{if } i \leq k \leq j, \\ j(i - k) & \text{if } i \geq k \geq j, \\ k(i + j - 2k) & \text{if } i, j \geq k, \end{cases}$$

and $\varphi(P_k, Q_k) = k^4$.

Hence it suffices to present an instance of a chain between P_k and Q_k having exactly such length. We do that in an algorithmic way, presenting a procedure to generate an order preserving path in the Hasse diagram of $\mathscr{A}(2k, k)$.

Procedure (Switch(t, r)). $1 \leq t, r \leq 2k - 1$.

Input: $A = (a_{ij}) \in \mathscr{A}(2k, k)$ such that the submatrix

$$\begin{pmatrix} a_{t,r} & a_{t,r+1} \\ a_{t+1,r} & a_{t+1,r+1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Output: $B = (b_{ij}) \in \mathscr{A}(2k, k)$ such that $b_{ij} = a_{ij}$ if $1 \leq i, j \leq 2k$ and $(i, j) \notin \{(t, r), (t, r + 1), (t + 1, r), (t + 1, r + 1)\}$, and

$$\begin{pmatrix} b_{t,r} & b_{t,r+1} \\ b_{t+1,r} & b_{t+1,r+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

It is easy to see that executing procedure Switch(t, r) the output covers the input in the Bruhat order for any choice of parameters t and r.

Our chain will be made by repeated applications of the procedure Switch(t, r).

Procedure (Switch–rows(t)). $1 \leq t \leq 2k - 1$.

Input: $A = (a_{ij}) \in \mathscr{A}(2k, k)$ such that rows $t, t + 1$ equal

$$\begin{pmatrix} 1, \dots, 1, 0, \dots, 0 \\ 0, \dots, 0, 1, \dots, 1 \end{pmatrix}.$$

For $\alpha = k$ down to 1 do
  Begin
    For $\beta = \alpha$ to $\alpha + k - 1$ do Switch(t, β).
  End.

Output: $B = (b_{ij}) \in \mathscr{A}(2k, k)$ such that $b_{ij} = a_{ij}$ for any $1 \leq i, j \leq 2k$ such that $i \neq t, t+1$, and rows $t, t+1$ equal

$$\begin{pmatrix} 0, \ldots, 0, 1, \ldots, 1 \\ 1, \ldots, 1, 0, \ldots, 0 \end{pmatrix}.$$

Algorithm 2.1 (Chain(k)). $0 \neq k \in \mathbb{N}$.

Input: P_k.
For $\alpha = k$ down to 1 do
  Begin
    For $\beta = \alpha$ to $\alpha + k - 1$ do Switch–rows(β).
  End.
Output: Q_k.

We can see that, for any choice of parameters, the procedure "Switch" is invoked k^2 times by procedure "Switch–rows", and that algorithm "Chain" recalls procedure "Switch–rows" k^2 times as well, so there are k^4 application of procedure "Switch". Since, as already remarked, each time that procedure "Switch" is recalled we are moving up (by one cover relation) in the Hasse diagram of the poset $(\mathscr{A}(2k, k), \preccurlyeq)$, all the constructed elements are pairwise distinct members of the desired chain. As a consequence we obtain our result.

Theorem 2.2. *For any positive integer k, the maximal length of a chain in the Bruhat order in $\mathscr{A}(2k, k)$ equals k^4.*

For the sake of clarity, we present in detail our construction of the chain for the case $k = 2$.

Example 2.1. $k = 2$.

$$\begin{pmatrix} 1&1&0&0 \\ 1&1&0&0 \\ 0&0&1&1 \\ 0&0&1&1 \end{pmatrix} \longmapsto \begin{pmatrix} 1&1&0&0 \\ 1&0&1&0 \\ 0&1&0&1 \\ 0&0&1&1 \end{pmatrix} \longmapsto \begin{pmatrix} 1&1&0&0 \\ 1&0&0&1 \\ 0&1&1&0 \\ 0&0&1&1 \end{pmatrix}$$

$$\longmapsto \begin{pmatrix} 1&1&0&0 \\ 0&1&0&1 \\ 1&0&1&0 \\ 0&0&1&1 \end{pmatrix} \longmapsto \begin{pmatrix} 1&1&0&0 \\ 0&0&1&1 \\ 1&1&0&0 \\ 0&0&1&1 \end{pmatrix} \longmapsto \begin{pmatrix} 1&1&0&0 \\ 0&0&1&1 \\ 1&0&1&0 \\ 0&1&0&1 \end{pmatrix}$$

$$\longmapsto \begin{pmatrix} 1&1&0&0 \\ 0&0&1&1 \\ 1&0&0&1 \\ 0&1&1&0 \end{pmatrix} \longmapsto \begin{pmatrix} 1&1&0&0 \\ 0&0&1&1 \\ 0&1&0&1 \\ 1&0&1&0 \end{pmatrix} \longmapsto \begin{pmatrix} 1&1&0&0 \\ 0&0&1&1 \\ 0&0&1&1 \\ 1&1&0&0 \end{pmatrix}$$

$$\longmapsto \begin{pmatrix} 1&0&1&0 \\ 0&1&0&1 \\ 0&0&1&1 \\ 1&1&0&0 \end{pmatrix} \longmapsto \begin{pmatrix} 1&0&0&1 \\ 0&1&1&0 \\ 0&0&1&1 \\ 1&1&0&0 \end{pmatrix} \longmapsto \begin{pmatrix} 0&1&0&1 \\ 1&0&1&0 \\ 0&0&1&1 \\ 1&1&0&0 \end{pmatrix}$$

$$\longmapsto \begin{pmatrix} 0&0&1&1\\ 1&1&0&0\\ 0&0&1&1\\ 1&1&0&0 \end{pmatrix} \longmapsto \begin{pmatrix} 0&0&1&1\\ 1&0&1&0\\ 0&1&0&1\\ 1&1&0&0 \end{pmatrix} \longmapsto \begin{pmatrix} 0&0&1&1\\ 1&0&0&1\\ 0&1&1&0\\ 1&1&0&0 \end{pmatrix}$$

$$\longmapsto \begin{pmatrix} 0&0&1&1\\ 0&1&0&1\\ 1&0&1&0\\ 1&1&0&0 \end{pmatrix} \longmapsto \begin{pmatrix} 0&0&1&1\\ 0&0&1&1\\ 1&1&0&0\\ 1&1&0&0 \end{pmatrix}.$$

References

[1] A. Barvinok, On the number of matrices and a random matrix with prescribed row and column sums and 0–1 entries, Adv. Math. 224 (1) (2010) 316–339.
[2] A. Berliner, R.A. Brualdi, L. Deaett, K.P. Kiernan, M. Schroeder, Row and column orthogonal (0, 1)-matrices, Linear Algebra Appl. 429 (11–12) (2008) 2732–2745.
[3] R.A. Brualdi, Combinatorial Matrix Classes, Encyclopedia of Mathematics and its Applications, vol. 108, Cambridge University Press, Cambridge, 2006.
[4] R.A. Brualdi, Algorithms for constructing (0, 1)-matrices with prescribed row and column sum vectors, Discrete Math. 306 (23) (2006) 3054–3062.
[5] R.A. Brualdi, Matrices of zeros and ones with fixed row and column sum vectors, Linear Algebra Appl. 33 (1980) 159–231.
[6] R.A. Brualdi, G. Dahl, The Bruhat shadow of a permutation matrix, Mathematical Papers in Honour of E. Marques de Sá, Textos Mat. Sér. B, 39, Univ. Coimbra, Coimbra, 2006, pp. 25–38.
[7] R.A. Brualdi, L. Deaett, More on the Bruhat order for (0, 1)-matrices, Linear Algebra Appl. 421 (2–3) (2007) 219–232.
[8] R.A. Brualdi, S.G. Hwang, A Bruhat order for the class of (0, 1)-matrices with row sum vector *R* and column sum vector *S*, Electron J. Linear Algebra 12 (2004/2005) 6–16.
[9] E.R. Canfield, B.D. McKay, Asymptotic enumeration of dense 0–1 matrices with equal row sums and equal column sums, Electron. J. Combin. 12 (2005) (Research Paper 29, 31 pp. (electronic)).
[10] J.A. Dias da Silva, A. Fonseca, Constructing integral matrices with given line sums, Linear Algebra Appl. 431 (9) (2009) 1553–1563.
[11] C.M. da Fonseca, R. Mamede, On (0, 1)-matrices with prescribed row and column sum vectors, Discrete Math. 309 (8) (2009) 2519–2527.
[12] D. Gale, A theorem on flows in networks, Pacific J. Math. 7 (1957) 1073–1082.
[13] C. Greenhill, B.D. McKay, Asymptotic enumeration of sparse nonnegative integer matrices with specified row and column sums, Adv. Appl. Math. 41 (4) (2008) 459–481.
[14] C. Greenhill, B.D. McKay, X. Wang, Asymptotic enumeration of sparse 0–1 matrices with irregular row and column sums, J. Combin. Theory Ser. A 113 (2) (2006) 291–324.
[15] M. Krause, A simple proof of the Gale–Ryser theorem, Amer. Math. Monthly 103 (1996) 335–337.
[16] B.D. McKay, X. Wang, Asymptotic enumeration of 0–1 matrices with equal row sums and equal column sums, Linear Algebra Appl. 373 (2003) 273–287.
[17] H.J. Ryser, Combinatorial properties of matrices of zeros and ones, Canad. J. Math. 9 (1957) 371–377.
[18] N.J.A. Sloane, The on-line encyclopedia of integer sequences, Notices Amer. Math. Soc. 50 (8) (2003) 912–915.
[19] http://www.oeis.org/A058527.